# **University of Calgary**

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# **CPSC 313: Introduction to Computability, Winter 2018**

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# **Assignment #2**

**For**

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**By**

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**1. Suppose that**

**ω = σ1σ2 . . . σn**

**is a string in L with positive length n such that σ1 = a (so that ω begins with an “a”), and the prefix σ1σ2 . . . σm does not belong to L, for every integer m such that 1 ≤ m ≤ n − 1.**

**(a) After noticing how the value of f changes whenever one more symbol is added, prove that f(σ1σ2 . . . σm) > 0 for every integer m such that 1 ≤ m ≤ n − 1.**

To prove f(σ1 σ2 . . . σm) > 0 for every integer m such that 1 ≤ m ≤ n − 1, we will use mathematical induction.

**Proof:**

**Choice of break point:** Since 1 ≤ m ≤ n − 1, we let β = 1. Then the case k = 1 will be considered in the basis.

**Basis:** If m = 1 then ω = σ1. Since σ1 = a as defined above, so that ω = a and f(a) = (number of a’s) - (number of b’s) = 1 - 0 = 1 in this case in which is true because 1 > 0.

**Inductive Step:** Let k be an arbitrary chosen integer such that k ≥ 1. It is necessary and sufficient to use the following:

*Inductive hypothesis:* f(σ1 σ2 . . . σi) > 0 for every integer i such that 1 ≤ i ≤ k, in which k < n − 1. To prove the following.

*Inductive claim:* f(σ1σ2 . . . σk+1) > 0

Since k ≥ 1, k + 1 ≥ 2 and k is integer between 1 and k, so that,

f(σ1 σ2 . . . σk+1) = f(σ1 σ2 . . . σk) + f(σk+1)

*First case:* when σk+1 = a, then f(σ1 σ2 . . . σk) + f(σk+1) = f(σ1 σ2 . . . σk) + 1. Since f(σ1 σ2 . . . σk) > 0 (*Inductive hypothesis*), f(σ1 σ2 . . . σk) + 1 > 1 in which satisfies the claim that f(σ1σ2 . . . σk+1) > 0.

*Second case:* when σk+1 = b, then f(σ1 σ2 . . . σk) + f(σk+1) = f(σ1 σ2 . . . σk) - 1. This leads to two cases:

① f(σ1 σ2 . . . σk) > 1:

In this case, f(σ1 σ2 . . . σk) > 1 ⇒ f(σ1 σ2 . . . σk) - 1 > 1 - 1 ⇒ f(σ1 σ2 . . . σk) -1 > 0, in which satisfies the claim that f(σ1σ2 . . . σk+1) > 0.

② f(σ1 σ2 . . . σk) = 1:

In this case, (number of a’s) = (number of b’s) + 1

Since σk+1 = b, f(σ1 σ2 . . . σk) + f(σk+1) = f(σ1 σ2 . . . σk) - 1. It means that (number of a’s) = (number of b’s) since there is one ‘b’ added. However, in order to let ω ⋲ L, this can only be true if ω = σ1 σ2 . . . σk+1 = σ1 σ2 . . . σn. By the assumption in *inductive hypothesis*, k < n − 1 ⇒ k + 1 < n. This leads to a contradiction since k + 1 ≠ n. So that, f(σ1 σ2 . . . σk) ≠ 1. Whenever σk+1 = b, f(σ1 σ2 . . . σk) > 1 (case ①) ⇒ f(σ1 σ2 . . . σk+1) > 0.

**Conclusion:** As a result, we have proved that f(σ1 σ2 . . . σm) > 0 for every integer m such that 1 ≤ m ≤ n − 1 by using mathematical induction.

**(b) Use this to prove that σn = b, that is, ω must end with a “b”.**

To prove σn = b, that is, ω must end with a ‘b’, we use contradiction.

**Proof:**

Let k be an arbitrary integer such that for all string σ1 σ2 . . . σk, such that 1 ≤ k ≤ n - 1. Since we have proved that f(σ1 σ2 . . . σm) > 0 for every integer m such that 1 ≤ m ≤ n - 1 by using mathematical induction, it is sufficient to show that f(σ1 σ2 . . . σk) > 0. Suppose that σn = b, in another words, σn = a since the symbols of the language ∑ only consists of ‘a’ and ‘b’. Since we know the string σ1σ2 . . . σn is a string in L with positive length n such that σn = a, we can then conclude that the string f(σ1σ2 . . . σn) = 0 since the number of a’s is equal to the number of b’s. Also, f(σ1σ2 . . . σn-1) = 0 - 1 = -1 since one ‘a’ has been removed from the string. However, it contradicts with the condition that f(σ1 σ2 . . . σm) > 0 for every integer m such that 1 ≤ m ≤ n - 1. As a result, a contradiction has been found and ω must end with a ‘b’.

**2. Suppose, next, that**

**ω = σ1σ2 . . . σn**

**is a string in *L* with positive length *n* such that there is an integer *m* such that and the prefix σ1σ2 . . . σm of *ω* with length *m* belongs to *L***

**Prove that the corresponding suffix σm+1 σm+2 . . . σn of *ω*, with length , is a string in \* that belongs to *L* as well.**

To prove the corresponding suffix σm+1 σm+2 . . . σn of *ω*, with length , is a string in \* that belongs to *L* as well, we first need to recognize the value for f(σ1σ2 . . . σn).

**Proof:**

The string σ1σ2 . . . σn is a string in L which means f(σ1σ2 . . . σn) = 0, and it’s prefix σ1σ2 . . . σm is a string in L which means f(σ1σ2 . . . σm) = 0 as well. Since ⇒ ⇒ ⇒ , we can write as the following:

f(σ1σ2 . . . σn) = f(σ1σ2 . . . σm) + f(σm+1 σm+2 . . . σn)

f(σm+1 σm+2 . . . σn) = f(σ1σ2 . . . σn) - f(σ1σ2 . . . σm)

f(σm+1 σm+2 . . . σn) = 0 - 0

f(σm+1 σm+2 . . . σn) = 0

Since f(σm+1 σm+2 . . . σn) = 0, this implies the number of a’s minus the number of b’s is zero. As a result, we have proved the corresponding suffix σm+1 σm+2 . . . σn of *ω*, with length n - m, is a string in \* that belongs to *L*.

**3. Using the above information1, describe a context-free grammar**

**(using the same alphabet ) such that *L(G) = L.***

Since we know the string belongs to L if and only if the numbers of a is equal to the numbers of b. Another way to define this language is to give a recursive definition - which also involves several other languages:

* Lab - the language that starts with an ‘a’ and ends with a ‘b’,
* Lba - the language that starts with a ‘b’ and ends with an ‘a’,
* Laa - the language that starts with an ‘a’ and ends with an ‘a’ as well, and
* Lbb - the language that starts with a ‘b’ and ends with a ‘b’ as well.

These sets can be defined using the following rules.

AB1. If then

BA1. If then

AA1. If then

BB1. If then

F1.

Let where

* Σ = {a, b};
* V = {S}. A context-free grammar with this variable is being designed such that, for all ,

i. if and only if .

* S = {S}. Since the desired “language of this grammar” is L, S is the start variable.
* R include rules that are developed as follows

- Rule #AB1 of the inductive definition is used to generate a rule to be included in R

- Rule #BA1 of the inductive definition is used to generate a rule to be included in R

- Rule #AA1 of the inductive definition is used to generate a rule to be included in R

- Rule #BB1 of the inductive definition is used to generate a rule to be included in R

- Rule #F1 of the inductive definition is used to generate a rule to be included in R

Since all of the rules in the inductive definition have been processed, it follows that R includes the following rule.

**4. Prove that , for the language \* defined in this assignment and for the grammar *G* that you have given when solving the above problem.**

In order to prove that is it necessary and sufficient to prove that can be derived from (using the rules in ) for every string .

This problem can be accomplished through induction.

**Claim:** For all strings \*

**Proof:**

By induction on The strong form of mathematical induction will be used.

**Basis:** Suppose is a string in with length zero. Then , the empty string.

Since is a rule in

(using the rule )

as required to establish the claim in this case.

**Inductive Step:** Let be an integer such that . It is necessary and sufficient to use the following

*Inductive hypothesis:* For every integer such that and all strings with length h, \* . To prove the following

*Inductive claim:* For all strings with length \*

With that, let be a string in with length . Then must be even, because does not contain any strings whose length are odd, so Then, where because this is a form of a string in the length in

Since and is a string with the length in , it follows by the *Inductive hypothesis* that \* .

Now

(using the rule )

\* (since \* by the Inductive Hypothesis)

Since is an arbitrarily chosen string with length in the result now follows by induction on the length of

Also, where because this is a form of a string in the length in

Since and is a string with the length in it follows by the inductive hypothesis that \*

Now

(using the rule )

\* (since \* by the Inductive Hypothesis)

Since is an arbitrarily chosen string with length in the result now follows by induction on the length of

Also, where because this is a form of a string in the length in

Since and both and are both strings with the length in it follows by the inductive hypothesis that \* and \*

Now

(using the rule )

\* (using the rules and )

\* (since \* by the Inductive Hypothesis)

\* (since \* by the Inductive Hypothesis)

Since is an arbitrarily chosen string with length in the result now follows by induction on the length of

Finaly, , where because this is a form of a string in the length in

Since and and are both strings with the length in it follows by the inductive hypothesis that \* and \*

Now

(using the rule )

\* (using the rules and )

\* (since \* by the Inductive Hypothesis)

\* (since \* by the Inductive Hypothesis)

Since is an arbitrarily chosen string with length in the result now follows by induction on the length of .

**Conclusion:**

Since is the starting variable of this grammar, it follows that

**5. Prove that , for the language \* defined in this assignment and for the grammar *G* that you have given above.**

**Claim:** For every string , if then .

**Proof:**

By induction on the length of a derivation of from . The standard form of mathematical induction can be used but, since there are no derivations of a string in from with length zero, the case that the length is one will be considered in the basis.

**Basis:** Let such that there is a derivation of from with length one.

There are four rules in . The derivation of from cannot begin with a use of the rule or or , because this could only produce the string , and this is not a string in .

The derivation of from must, therefore, begin with a use of rule , so this derivation has the form

(using the rule )

Thus . Since and , it follows that in this case.

**Inductive Step:** Let k be an integer such that . It is necessary and sufficient to use the following

*Inductive hypothesis:*For every string , if there is a derivation of from with length , then . To prove the following

*Inductive claim:*For every string , if there is a derivation of from with length , then .

Suppose such that there is a derivation of from with length .

Since , so that this derivation cannot begin with a use of the rule . There are no variables on the right-hand side of this rule, so it can only begin a derivation with length one.

Since there are three rules that can be applied:

The derivation must begin an application of the rule , so it has the form

(using the rule )

(for \*)

where and there is a derivation with length .

Also another derivation must begin an application of the rule , so it has the form

(using the rule )

(for \*)

where and there is a derivation with length .

Knowing that and from above, then it follows for following.

The derivation must begin an application of the rule , so it has the form

(using the rule )

(using the rule )

(using the rule )

(for )

(for )

where and there is a derivation with length .

Also another derivation must begin an application of the rule , so it has the form

(using the rule )

(using the rule )

(using the rule )

(for )

(for )

where and there is a derivation with length .

It follows by the *Inductive hypothesis* that , so that and and and for some natural number ℕ. However, it now follows by above that and and and , so that as well.

Since was an arbitrarily chosen element of such that there is a derivation of from with length , this establishes the *Inductive Claim*, as needed to complete the inductive step, and the proof of the claim.

It now follows that **.**

**Conclusion:**

Since it has been shown that ,, so that is the language of the grammar .